

Relative entropy applied to the stability of viscous shocks up to a translation for scalar conservation laws

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Abstract

We consider inviscid limits to shocks for viscous scalar conservation laws in one space dimension, with strict convex fluxes. We show that we can obtain sharp estimates in L^2 , for a class of large perturbations. Those perturbations can be chosen big enough to destroy the viscous layer. This shows that the fast convergence to the shock does not depend on the fine structure of the viscous layers. This is the first application of the relative entropy method developed in [22], [23] to the study of an asymptotic limit to a shock.

Keywords: viscous scalar conservation laws; inviscid limits; stability; relative entropy method; shocks.

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1 Introduction and the main result

For any strictly convex flux function $A \in C^2(\mathbb{R})$, we consider the family of viscous scalar conservation laws in one space dimension:

$$\begin{cases} \partial_t U + \partial_x A(U) = \varepsilon \partial_{xx}^2 U & \text{for } t > 0, x \in \mathbb{R}, \\ U(0, x) = U_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1)$$

for any $\varepsilon > 0$ and $U_0 \in L^\infty$. Global unique solutions to (1) have been constructed by Hopf [17] and Oleřnik [28]. The inviscid case, $\varepsilon = 0$, is covered by the theory of Kruřkov [20]. Kuznetsov showed in [21] that, for fixed initial

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values U_0 , the solutions of (1) converges, when ε goes to zero, to the solution of the inviscid scalar conservation law (the equation (1) with $\varepsilon = 0$). He showed also that, for general initial data, the optimal rate of convergence in L^1 is $\sqrt{\varepsilon}$. In the case of the convergence to a shock, however, the rate is better. This is, usually, linked to the formation of layers.

In this paper we consider the asymptotic limit for general initial values. We are particularly interested in the cases where the initial values carry too much entropy for the structure of the layer to be preserved asymptotically. The shocks solutions of the inviscid case ($\varepsilon = 0$) can be described as follows. Consider two constants $C_L > C_R$, and the associated function defined by

$$S_0(x) = \begin{cases} C_L & \text{if } x < 0, \\ C_R & \text{if } x \geq 0. \end{cases} \quad (2)$$

Then, the Rankine-Hugoniot conditions ensures that the function

$$S_0(x - \sigma t) \quad \text{with} \quad \sigma := \frac{A(C_L) - A(C_R)}{C_L - C_R}, \quad (3)$$

is a solution to the inviscid equation (1) with $\varepsilon = 0$. The condition $C_L > C_R$ implies that they verify the entropy conditions, that is:

$$\partial_t \eta(U) + \partial_x G(U) \leq 0, \quad t > 0, \quad x \in \mathbb{R},$$

for any convex functions η , and $G' = \eta' A'$.

Our main result is the following.

Theorem 1.1. *Let $C_L > C_R$ and $U_0 \in L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \quad \text{and} \quad \left(\frac{d}{dx} U_0\right)_+ \in L^2(\mathbb{R}).$$

Then, for any $T > 0$, there exist $\varepsilon_0 > 0$ and $C^ > 0$ such that the followings hold true.*

I. For any U solution to (1) with $0 < \varepsilon \leq \varepsilon_0$, there exists a curve $X \in L^\infty(0, T)$ such that $X(0) = 0$ and for any $0 < t < T$:

$$\|U(t) - S(t)\|_{L^2(\mathbb{R})}^2 \leq \|U_0 - S_0\|_{L^2(\mathbb{R})}^2 + C^* \varepsilon \log(1/\varepsilon), \quad (4)$$

where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (2).

II. Moreover, this curve satisfies

$$|\dot{X}(t)| \leq C^* \quad \text{and} \quad (5)$$

$$|X(t) - \sigma t|^2 \leq C^* t^{2/3} \left(\|U_0 - S_0\|_{L^2(\mathbb{R})}^2 + \varepsilon \log(1/\varepsilon) \right). \quad (6)$$

III. The constant ε_0 depends only on $\|(\frac{d}{dx}U_0)_+\|_{L^2}$, C_L , C_R , $\|U_0\|_{L^\infty}$ and the flux function A , while C^* depends only on the same set as well as T .

Remark 1.1. Note that our estimates do not depend on any local BV norms of U_0 . The assumption $U_0 \in BV_{loc}$ ensures that $\frac{d}{dx}U_0$ is a Radon measure. Hence, $(\frac{d}{dx}U_0)_+$ is also a Radon measure, and the condition $(\frac{d}{dx}U_0)_+ \in L^2$ makes sense.

This result shows a rate of convergence slightly worse than ε (to the log), for the inviscid limit to a shock, measured via the L^2 norm (squared). In the case of the limit to a regular solution of the inviscid case, the rate of convergence is $\sqrt{\varepsilon}$ (see [32], for instance). We also refer to [21], Goodman and Xin [16], Bressan, Liu and Yang [6], Lewicka [24], Bressan and Yang [5], Christoforou and Trivisa [10].

An easy layer study shows that ε is the optimal rate for shocks with special initial data. Indeed, one can construct an associated steady viscous layer (see for example Il'in and Oleĭnik [18]) S_1 solution to

$$\begin{cases} A(S_1) - A(C_L) - \sigma(S_1 - C_L) = S_1', & x \in \mathbb{R}, \\ \lim_{x \rightarrow -\infty} S_1 = C_L, & \lim_{x \rightarrow +\infty} S_1 = C_R. \end{cases} \quad (7)$$

It is easy to show that $S_1((x - \sigma t)/\varepsilon)$ is a solution to (1) with initial data $S_1(x/\varepsilon)$. In this case, the rate of convergence is of order ε since:

$$\int_{\mathbb{R}} |S_1((x - \sigma t)/\varepsilon) - S_0(x - \sigma t)|^2 dx \leq \varepsilon \int_{\mathbb{R}} |S_1(x) - S_0(x)|^2 dx = C\varepsilon.$$

This layer study can be extended to the case of small initial perturbation where:

$$\int_{\mathbb{R}} |U_0(x) - S_0(x)|^p dx \leq C\varepsilon,$$

for a $1 \leq p < \infty$. In this case, for a solution U to (1), we can consider

$$V(t, x) = U(\varepsilon t, \varepsilon x),$$

and study the asymptotic for large time. The function V is a solution to the equation

$$\begin{cases} \partial_t V + \partial_x A(V) - \partial_{xx}^2 V = 0, \\ V(0, x) = U(0, x/\varepsilon). \end{cases}$$

The convergence to S_1 , up to a (constant) drift, in this setting, has been extensively studied (see for instance [18], Freistühler and Serre [14], Kenig and Merle [19]). In this situation of small perturbation of the initial shock, those results show that the convergence with rate ε for the system (1) is due to the asymptotic limit in large time of the layer function $U(\cdot/\varepsilon)$ to $S_1((\cdot - \sigma t)/\varepsilon)$.

This layer study, however, collapses when

$$\int_{\mathbb{R}} |U_0(x) - S_0(x)|^2 dx \gg \varepsilon.$$

In this situation, there is too much entropy for the asymptotic limit of the layer structure to be true. The physical layer may be destroyed. Theorem 1.1 shows that, nevertheless, the sharp convergence (up to the log) still holds.

Taking a limit as ε goes to 0 in Theorem 1.1, we recover the L^2 stability of shocks (up to a drift) first showed by Leger in [22]. Note that the stability result has to be up to a drift which depends on the solution itself (and may be not unique). This feature is also true for our result. The drift cannot be taken constant, as in the case of the layer problem.

Our result is based on the relative entropy method first used by Dafermos and DiPerna to show L^2 stability and uniqueness of Lipschitzian solutions to conservation laws [11, 12, 13]. They showed, in particular, that if \bar{U} is a Lipschitzian solution of a suitable conservation law on a lapse of time $[0, T]$, then for any bounded weak entropic solution U it holds:

$$\int_{\mathbb{R}} |U(t) - \bar{U}(t)|^2 dx \leq C \int_{\mathbb{R}} |U(0) - \bar{U}(0)|^2 dx, \quad (8)$$

for a constant C depending on \bar{U} and T .

The relative entropy method is also an important tool in the study of asymptotic limits. The main idea is that convergence holds thanks to the strong stability of the solutions of the limit equations. Roughly speaking, if we have good consistency of ε models, with respect to the limit one, then non linearities are driven by the strong stability of the solution of the limit equation. Applications of the relative entropy method in this context began with the work of Yau [33] and have been studied by many others. For incompressible limits, see Bardos, Golse, Levermore [1, 2], Lions and Masmoudi [25], Saint Raymond et al. [15, 30, 26, 29]. For compressible models, see Tzavaras [31] in the context of relaxation and [4, 3, 27] in the context of hydrodynamical limits. However, in all those cases, the method works as long as the limit solution is Lipschitz. This is due to the fact that strong stability as (8) is not true when \bar{U} has a discontinuity. It has been proven in [22, 23], however, that some shocks are strongly stable up to a shift (see also related works from Chen and Frid [7, 8] and Chen, Frid and Li [9]). This article is the first extension of those results of stability, to the study of asymptotic limits to a shock. This is a part of the program initiated in [32].

The result can be extended to any entropy in the following way. Fix any strictly convex function $\eta \in C^2$ as an entropy. We define the associated relative entropy functional $\eta(\cdot|\cdot)$ as

$$\eta(x|y) := \eta(x) - \eta(y) - \eta'(y)(x - y).$$

We then have the following extension.

Theorem 1.2. *Consider a strictly convex entropy functional $\eta \in C^2(\mathbb{R})$. Let $C_L > C_R$ and $U_0 \in L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \quad \text{and} \quad \left(\frac{d}{dx}U_0\right)_+ \in L^2(\mathbb{R}).$$

Then, for every $T > 0$, there exist $\varepsilon_0 > 0$ and $C^ > 0$ such that the followings hold true.*

- I. *For any U solution to (1) with $0 < \varepsilon \leq \varepsilon_0$, there exists a curve $X \in L^\infty(0, T)$ such that $X(0) = 0$, and for any $0 < t < T$, and for any α verifying $\varepsilon \leq \alpha \leq \varepsilon_0$, we have:*

$$\int_{\{|x-X(t)| \geq C^*\alpha\}} \eta(U(t, x)|S(t, x)) dx \leq \int_{\mathbb{R}} \eta(U_0(x)|S_0(x)) dx + C^* e^{-\alpha/\varepsilon}, \quad (9)$$

where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (2).

- II. *Moreover, this curve satisfies*

$$|\dot{X}(t)| \leq C^* \quad \text{and} \quad (10)$$

$$|X(t) - \sigma t|^2 \leq C^* t^{2/3} \left(\int_{\mathbb{R}} \eta(U_0(x)|S_0(x)) dx + \varepsilon \log(1/\varepsilon) \right). \quad (11)$$

- III. *The constant ε_0 depends only on $\|(\frac{d}{dx}U_0)_+\|_{L^2}$, C_L , C_R , $\|U_0\|_{L^\infty}$, the flux function A and the entropy functional η , while C^* depends only on the same set as well as T .*

Theorem 1.1 is a direct application of Theorem 1.2 with $\eta(x) := x^2$, and $\alpha = \varepsilon \log(1/\varepsilon)$. Indeed, in this case we have $\eta(x|y) = (x - y)^2$, and

$$\int_{\{|x-X(t)| \leq C^*\alpha\}} \eta(U(t, x)|S(t, x)) dx \leq C |\{|x - X(t)| \leq C^*\alpha\}| \leq CC^*\alpha.$$

For the rest of the paper, we will assume that the initial value U_0 lies not only BV_{loc} but also C^1 . It allows us to work with smooth solutions $U \in C^1([0, T] \times \mathbb{R})$. The general BV_{loc} case can be obtained by a density argument.

The idea of the proof is to study the evolution of the relative entropy of the solution with respect to the shock, outside of a small region centered at $X(t)$ (this small region corresponds to the layer localization):

$$\int_{-\infty}^{X(t)-\delta\varepsilon} \eta(U(t, x)|C_L) dx + \int_{X(t)+\delta\varepsilon}^{\infty} \eta(U(t, x)|C_R) dx. \quad (12)$$

The change in time involves two effects. One is due to the hyperbolic part of the equation, and the second involves the parabolic part (or order ε). In [22], it was shown that, for the hyperbolic case $\varepsilon = 0$, with zero layer width $\delta = 0$, this quantity is non-increasing when we choose wisely the drift $X(t)$. When considering the viscous term, the layer with width $\delta\varepsilon$ is introduced to mollify its effect. The idea is then that the stability induced by the hyperbolic part is enough to compensate the parabolic one, provided that we consider a layer fat enough. For technical considerations, we will fix $\delta = \log(1/\varepsilon)$. The drift $X(t)$ is still chosen with respect to the hyperbolic part of the equation in a similar way as in [22]. The stability effect is preserved, despite the non zero layer width, thanks to a monotonicity property induced in the layer by the additional assumption $(\frac{d}{dx}U_0)_+ \in L^2(\mathbb{R})$.

2 Evolution of the relative entropy

For $\delta > 0$, we consider a Lipschitz nondecreasing function ϕ_δ to localize the layer, verifying

$$\phi_\delta(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq \delta. \end{cases}$$

To get the optimal result, we will later fix a special function (see (19)).

For any fixed $\delta > 0$ and $X \in C^1([0, T])$, we are interesting in the evolution of

$$H(t) = \int_{-\infty}^{\infty} \phi_\delta^2(|x - X(t)|/\varepsilon) \eta(U(t, x)|S(t, x)) dx, \quad (13)$$

where the function S depends on X as in the statement of the theorems. A special value of δ (depending on ε), and of the function X will be chosen later. Note that $H(t)$ is controlling the quantity (12). From now on, we drop the subindex δ for ϕ_δ for a simple presentation.

Let us denote $F(\cdot, \cdot)$ the flux of the relative entropy $\eta(\cdot|\cdot)$ defined by

$$F(x, y) := G(x) - G(y) - \eta'(y)(A(x) - A(y)). \quad (14)$$

Note that

$$\begin{aligned} \partial_U \eta(U|C) &= \eta'(U) - \eta'(C), \\ \partial_U F(U, C) &= G'(U) - \eta'(C)A'(U) = (\eta'(U) - \eta'(C))A'(U). \end{aligned}$$

So, for any solution U of (1) and any constant C , multiplying (1) by $\eta'(U) - \eta'(C)$, we get

$$\partial_t \eta(U|C) + \partial_x F(U, C) = \varepsilon(\eta'(U) - \eta'(C))\partial_{xx}^2 U \quad (15)$$

We have the following lemma.

Lemma 2.1. *The function H , defined in (13), satisfies the following on $(0, T)$*

$$\begin{aligned}
\frac{dH}{dt}(t) &= \\
&\int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon} \phi\left(\frac{-x+X(t)}{\varepsilon}\right) \phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \left[\dot{X}(t)\eta(U(t,x)|C_L) - F(U(t,x), C_L)\right] dx \\
&+ \varepsilon \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 \partial_{xx}^2 U(t,x) (\eta'(U(t,x)) - \eta'(C_L)) dx \\
&- \int_{X(t)}^{X(t)+\delta\varepsilon} \frac{2}{\varepsilon} \phi\left(\frac{x-X(t)}{\varepsilon}\right) \phi'\left(\frac{x-X(t)}{\varepsilon}\right) \left[\dot{X}(t)\eta(U(t,x)|C_R) - F(U(t,x), C_R)\right] dx \\
&+ \varepsilon \int_{X(t)}^{\infty} \left[\phi\left(\frac{x-X(t)}{\varepsilon}\right)\right]^2 \partial_{xx}^2 U(t,x) (\eta'(U(t,x)) - \eta'(C_R)) dx \\
&:= (L)_{Hyp} + (L)_{Dif} + (R)_{Hyp} + (R)_{Dif}
\end{aligned}$$

Proof. First we split the above integral into the two parts:

$$\begin{aligned}
H(t) &= \int_{-\infty}^{\infty} \left[\phi\left(\frac{|x-X(t)|}{\varepsilon}\right)\right]^2 \eta(U|S) dx \\
&= \int_{-\infty}^{\infty} \left(\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 + \left[\phi\left(\frac{x-X(t)}{\varepsilon}\right)\right]^2\right) \eta(U|S) dx \\
&= \int_{-\infty}^{\infty} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 \eta(U|C_L) dx + \int_{-\infty}^{\infty} \left[\phi\left(\frac{x-X(t)}{\varepsilon}\right)\right]^2 \eta(U|C_R) dx \\
&:= H^L + H^R.
\end{aligned}$$

To compute $\frac{d}{dt}(H^L)$, we put $C = C_L$ in (15), multiply by $\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2$, and integrate in x . Then we have

$$\begin{aligned}
\frac{d}{dt}(H^L) &= \int_{-\infty}^{\infty} \partial_t \left(\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2\right) \eta(U|C_L) dx \\
&\quad + \int_{-\infty}^{\infty} \partial_x \left(\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2\right) F(U, C_L) dx \\
&\quad + \varepsilon \int_{-\infty}^{\infty} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 \partial_{xx}^2 U (\eta'(U) - \eta'(C_L)) dx \\
&= \int_{X(t)-\delta\varepsilon}^{X(t)} \left(\frac{2}{\varepsilon}\right) \cdot \phi\left(\frac{-x+X(t)}{\varepsilon}\right) \phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \left[\dot{X}(t)\eta(U|C_L) - F(U, C_L)\right] dx \\
&\quad + \varepsilon \cdot \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 \partial_{xx}^2 U (\eta'(U) - \eta'(C_L)) dx = (L)_{Hyp} + (L)_{Dif}
\end{aligned}$$

We get the result for $\frac{d}{dt}(H^R) = (R)_{Hyp} + (R)_{Dif}$ in the same way. \square

3 Study of the contribution of the hyperbolic structure

In this section, we show that by choosing a special drift function $X(\cdot)$, the hyperbolic effects become nonpositive. This will be used in section 4 to control the parabolic effects.

Following [22], we define the normalized relative entropy flux $f(\cdot, \cdot)$ by

$$f(x, y) := \frac{F(x, y)}{\eta(x|y)}.$$

We have the following properties.

Lemma 3.1. *For any $L > 0$, there exists a constant $\Lambda > 0$, such that for any x, y with $|x|, |y| \leq L$, we have*

$$\begin{aligned} 1/\Lambda &\leq \eta''(x) \leq \Lambda, \\ \frac{1}{2\Lambda}(x-y)^2 &\leq \eta(x|y) \leq \frac{1}{2}\Lambda(x-y)^2, \\ |F(x, y)| &\leq \Lambda(x-y)^2, \\ 0 &\leq (\partial_x f)(x, y) \leq \Lambda, \\ 1/\Lambda &\leq (\partial_y f)(x, y). \end{aligned} \tag{16}$$

The proof of this lemma can be found in [22].

We now define the shift function X . It is the solution of the following O.D.E.

$$\begin{cases} \dot{X}(t) = f\left(U(t, X(t)), \frac{C_L + C_R}{2}\right) \\ X(0) = 0 \end{cases} \tag{17}$$

Note that for any $\varepsilon > 0$, $U \in C^1([0, T] \times \mathbb{R})$ (since $U_0 \in C^1(\mathbb{R})$). The existence and uniqueness of X comes from the Cauchy-Lipschitz theorem.

First, X is Lipschitz, since we have from Lemma 3.1

$$|\dot{X}(t)| \leq \frac{\left|F\left(U(t, X(t)), \frac{C_L + C_R}{2}\right)\right|}{\eta\left(U(t, X(t)) \middle| \frac{C_L + C_R}{2}\right)} \leq 2\Lambda^2 \tag{18}$$

where we used the fact $\|U(t)\|_{L^\infty} \leq \|U_0\|_{L^\infty}$ for $t > 0$. It proves (10).

Note that from the definition of X , if the the velocity U were constant in the layer (that is $U(t, x) \sim U(t, X(t))$ for $-\delta\varepsilon \leq x - X(t) \leq \delta\varepsilon$), then, from the last property of Lemma 3.1, we would have directly that

$$(L)_{Hyp} + (R)_{Hyp} \leq -\frac{C_L - C_R}{\Lambda}(\eta(U(t, X(t))|C_L) + \eta(U(t, X(t))|C_R)).$$

However, this is too much to hope, since the layer characterize the region where the function $U(t, \cdot)$ is expected to drop from C_L to C_R . We still can show that the hyperbolic terms are negative, provided that the behavior of U in the layer is not too much oscillatory (the values can drop, but not much bounce back). This last property of U is proved in the following lemma which can be seen as a weak version of the Oleřnik's principle .

Lemma 3.2. $\|(\partial_x U(t))_+\|_{L^2(\mathbb{R})} \leq \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}$ for any $t > 0$.

Proof. We differentiate (1) w.r.t. x , multiply $(\partial_x U)_+$ and integrate in x to get

$$\begin{aligned} 0 &= \int (\partial_x U)_+ \left[\partial_t \partial_x U + A''(U) |\partial_x U|^2 + A'(U) \partial_{xx}^2 U - \varepsilon \partial_{xxx}^3 U \right] dx \\ &= \int \left[\frac{1}{2} \partial_t ((\partial_x U)_+)^2 + A''(U) (\partial_x U)_+^3 \right. \\ &\quad \left. + A'(U) \partial_x \left(\frac{[(\partial_x U)_+]^2}{2} \right) + \varepsilon |\partial_x ((\partial_x U)_+)|^2 \right] dx. \end{aligned}$$

Then, we use the integration by parts to get

$$\begin{aligned} &= \int \left[\frac{1}{2} \partial_t ((\partial_x U)_+)^2 + \frac{1}{2} A''(U) (\partial_x U)_+^3 + \varepsilon |\partial_x ((\partial_x U)_+)|^2 \right] dx \\ &\geq \frac{1}{2} \frac{d}{dt} \int [(\partial_x U)_+]^2 dx. \end{aligned}$$

□

We now prove the main proposition of this section.

Proposition 3.3. *Let $(L)_{Hyp}$ and $(R)_{Hyp}$ be such as in Lemma 2.1. There exists a constant $\theta > 0$ such that, for any ε, δ satisfying*

$$\varepsilon \delta \leq \theta,$$

we have

$$\begin{aligned} &(L)_{Hyp} + (R)_{Hyp} \\ &\leq -\frac{\theta}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \phi\left(\frac{|x-X(t)|}{\varepsilon}\right) \phi'\left(\frac{|x-X(t)|}{\varepsilon}\right) (U(t, x) - S(t, x))^2 dx. \end{aligned}$$

Proof. We use the definition of $X(t)$ to get

$$(L)_{Hyp} = \int_{X(t)-\delta\varepsilon}^{X(t)} \left(\frac{2}{\varepsilon}\right) \cdot \phi\left(\frac{-x+X(t)}{\varepsilon}\right) \cdot \phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \cdot \eta(U|C_L) \cdot h(t, x) dx$$

where $h(t, x) := \left[f\left(U(t, X(t)), \frac{C_L+C_R}{2}\right) - f(U(t, x), C_L) \right]$.

In order to make the function $h(t, x)$ strictly negative over the domain of the above integral, we use the condition $(\frac{d}{dx}U_0)_+ \in L^2(\mathbb{R})$. Indeed, we observe that, for any $x \in [X(t) - \delta\varepsilon, X(t)]$,

$$\begin{aligned} U(t, X(t)) - U(t, x) &= \int_x^{X(t)} (\partial_x U)(t, y) dy \leq \int_x^{X(t)} (\partial_x U)_+(t, y) dy \\ &\leq \|(\partial_x U(t))_+\|_{L^2(\mathbb{R})} \sqrt{|X(t) - x|} \leq \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})} \sqrt{\delta\varepsilon}, \end{aligned}$$

where we used that $\|(\partial_x U(t))_+\|_{L^2}$ is not increasing (see Lemma 3.2).

We can rewrite the function h as

$$\begin{aligned} h(t, x) &= f\left(U(t, X(t)), \frac{C_L + C_R}{2}\right) - f\left(U(t, x), \frac{C_L + C_R}{2}\right) \\ &\quad + f\left(U(t, x), \frac{C_L + C_R}{2}\right) - f(U(t, x), C_L). \end{aligned}$$

Since f is increasing with respect to the first variable, we have

$$\begin{aligned} h(t, x) &\leq f\left(U(t, x) + \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})} \sqrt{\delta\varepsilon}, \frac{C_L + C_R}{2}\right) - f\left(U(t, x), \frac{C_L + C_R}{2}\right) \\ &\quad + f\left(U(t, x), \frac{C_L + C_R}{2}\right) - f(U(t, x), C_L). \end{aligned}$$

Then, thanks to Lemma 3.1, we get

$$h(t, x) \leq \Lambda \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})} \sqrt{\delta\varepsilon} - \frac{C_L - C_R}{2\Lambda} \leq -\theta < 0$$

for $\sqrt{\delta\varepsilon}$ small enough.

Since $\phi(\cdot), \phi'(\cdot)$ and $\eta(\cdot|\cdot) \geq 0$, we get

$$(L)_{Hyp} \leq -\theta \int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon} \phi\left(\frac{-x + X(t)}{\varepsilon}\right) \phi'\left(\frac{-x + X(t)}{\varepsilon}\right) \eta(U|C_L) dx.$$

Then, from Lemma 3.1, we have (changing the constant θ if necessary)

$$(L)_{Hyp} \leq -\theta \int_{X(t)-\delta\varepsilon}^{X(t)} \left(\frac{2}{\varepsilon}\right) \phi\left(\frac{-x + X(t)}{\varepsilon}\right) \phi'\left(\frac{-x + X(t)}{\varepsilon}\right) (U - C_L)^2 dx.$$

In a similar way, we obtain the following estimate on $(II)_{Hyp}$.

$$(R)_{Hyp} \leq -\theta \int_{X(t)}^{X(t)+\delta\varepsilon} \left(\frac{2}{\varepsilon}\right) \phi\left(\frac{x - X(t)}{\varepsilon}\right) \phi'\left(\frac{x - X(t)}{\varepsilon}\right) (U - C_R)^2 dx.$$

Combining the two last inequalities gives the desired result. \square

4 Control of the parabolic terms

For any $\delta \geq 1/\theta$, we now fix the function ϕ_δ in the following explicit way.

$$\phi_\delta(x) = \begin{cases} \theta e^{1-\theta\delta}x, & \text{for } x \in [0, 1/\theta), \\ e^{\theta(x-\delta)}, & \text{for } x \in [1/\theta, \delta]. \end{cases} \quad (19)$$

It can be proved that this functional minimize

$$\int_0^\delta (\phi'(x))^2 \chi_{\{\phi' > \theta\phi\}} dx$$

among all increasing Lipschitz functions with $\phi(0) = 0$, $\phi(\delta) = 1$ and $\delta \geq 1/\theta$. We will only use the straightforward computation:

$$\int_0^\delta (\phi'_\delta(x))^2 \chi_{\{\phi'_\delta > \theta\phi_\delta\}} dx = C_\theta \cdot e^{-2\theta\delta}. \quad (20)$$

This section is dedicated to the proof of the following proposition.

Proposition 4.1. *There exists constants $\theta, C > 0$ such that for any ε, δ verifying*

$$\frac{1}{\theta} \leq \delta \quad \text{and} \quad \varepsilon\delta \leq \theta,$$

we have

$$\frac{dH(t)}{dt} \leq Ce^{-\theta\delta}.$$

Proof. First, we estimate the term $(L)_{Dif}$. Integrating by parts, we obtain

$$\begin{aligned} (L)_{Dif} &= \int_{-\infty}^{X(t)} 2\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \partial_x U (\eta'(U) - \eta'(C_L)) dx \\ &\quad - 2\varepsilon \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \eta''(U) |\partial_x U|^2 dx \end{aligned}$$

Then, using Hölder's inequality and Lemma 3.1, we get

$$\begin{aligned} (L)_{Dif} &\leq \frac{2\varepsilon}{\Lambda} \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 |\partial_x U|^2 dx \\ &\quad + \frac{\Lambda}{8\varepsilon} \int_{-\infty}^{X(t)} \left[2\phi'\left(\frac{-x+X(t)}{\varepsilon}\right) (\eta'(U) - \eta'(C_L)) \right]^2 dx \\ &\quad - \frac{2\varepsilon}{\Lambda} \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 |\partial_x U|^2 dx \\ &\leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)} \left[\phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 |U - C_L|^2 dx. \end{aligned}$$

In the same way, we obtain the following estimate for $(R)_{Dif}$

$$(R)_{Dif} \leq \frac{C}{\varepsilon} \int_{X(t)}^{X(t)+\delta\varepsilon} \left[\phi' \left(\frac{x - X(t)}{\varepsilon} \right) \right]^2 |U - C_R|^2 dx.$$

Combining the two last inequalities, we find

$$(L)_{Dif} + (R)_{Dif} \leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi' \left(\frac{|x - X(t)|}{\varepsilon} \right) \right]^2 |U(t, x) - S(t, x)|^2 dx. \quad (21)$$

Using Lemma 2.1, Proposition 3.3, and (21), we find

$$\frac{dH(t)}{dt} \leq \frac{1}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi' (C\phi' - \theta\phi) \right] \left(\frac{|x - X(t)|}{\varepsilon} \right) |U(t, x) - S(t, x)|^2 dx. \quad (22)$$

Using that $U - S$ is a bounded function, and doing the change of variables $z = (x - X(t))/\varepsilon$, we find:

$$\begin{aligned} \frac{dH(t)}{dt} &\leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[(\phi')^2 \chi_{\{C\phi' - \theta\phi > 0\}} \right] \left(\frac{|x - X(t)|}{\varepsilon} \right) |U(t, x) - S(t, x)|^2 dx \\ &\leq \frac{C \|U(t) - S(t)\|_{L^\infty}^2}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[(\phi')^2 \chi_{\{C\phi' - \theta\phi > 0\}} \right] \left(\frac{|x - X(t)|}{\varepsilon} \right) dx \\ &\leq C \int_0^\delta (\phi')^2(z) \chi_{\{C\phi' - \theta\phi > 0\}}(z) dz. \end{aligned}$$

Changing the constant θ if needed, and using (20), gives the desired result. \square

5 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Integrating the estimate of Proposition 4.1 between 0 and $t \in (0, T)$ gives the result of (I). Indeed, for any ε, δ with $\frac{1}{\theta} \leq \delta$ and $\varepsilon\delta \leq \theta$, where θ is the constant from Proposition 4.1, we have

$$\begin{aligned} \int_{\{|x - X(t)| \geq \delta\varepsilon\}} \eta(U(t, x)|S(t, x)) dx &\leq H(t) \leq H(0) + \int_0^t \frac{d}{ds} H(s) ds \\ &\leq \int_{\mathbb{R}} \eta(U_0|S_0) dx + CT e^{-\theta\delta} \end{aligned}$$

By taking $\varepsilon_0 := \theta^2$, we have for any $\varepsilon \leq \alpha \leq \varepsilon_0$,

$$\int_{\{|x - X(t)| \geq \alpha/\theta\}} \eta(U(t, x)|S(t, x)) dx \leq \int_{\mathbb{R}} \eta(U_0|S_0) dx + CT e^{-\alpha/\varepsilon}.$$

It proves our main estimate (9) by taking C^* large enough.

Observe that

$$\int_{\mathbb{R}} \eta(U|S) dx = \int_{\{|x-X(t)| \geq C^* \alpha\}} \eta(U|S) dx + \int_{\{|x-X(t)| < C^* \alpha\}} \eta(U|S) dx$$

and the second term is bounded by $CC^*\alpha$. Thus, by taking $\alpha = \varepsilon \log(1/\varepsilon)$, we obtain for any $t \in (0, T)$,

$$\int_{\mathbb{R}} \eta(U|S) dx \leq \int_{\mathbb{R}} \eta(U_0|S_0) dx + C^* \varepsilon \log(1/\varepsilon) \quad (23)$$

for any $\varepsilon \leq \varepsilon_0$ (changing ε_0 and C^* if needed).

It only remains to prove (11). We define first $\psi(x) := \begin{cases} 0 & \text{if } |x| > 2, \\ 1 & \text{if } |x| \leq 1 \\ 2 - |x| & \text{if } 1 < |x| \leq 2 \end{cases}$.

Let $s \in (0, t)$ and $R > 0$. We multiply $\Psi_R(s, x) := \psi(\frac{x-X(s)}{R})$ to the equation (1) and integrate in x to get

$$\begin{aligned} 0 &= -\frac{d}{ds} \int \Psi_R \cdot U dx + \int \partial_x(\Psi_R) A(U) dx + \int \partial_t(\Psi_R) U dx + \varepsilon \int \Psi_R \cdot \partial_{xx}^2 U dx \\ &= -\underbrace{\frac{d}{ds} \int \psi\left(\frac{x-X(s)}{R}\right) \cdot U(s, x) dx}_{(I)} \\ &\quad + \underbrace{\frac{1}{R} \int \psi'\left(\frac{x-X(s)}{R}\right) \cdot \left(A(U(s, x)) - \dot{X}(s)U(s, x)\right) dx}_{(II)} \\ &\quad - \underbrace{\varepsilon \frac{1}{R} \int \psi'\left(\frac{x-X(s)}{R}\right) \cdot \partial_x U(s, x) dx}_{(III)}. \end{aligned}$$

By using the above observation, we have

$$\begin{aligned} (\sigma - \dot{X}(s)) &= \frac{1}{C_L - C_R} \left(A(C_L) - A(C_R) - (C_L - C_R) \dot{X}(s) \right) \\ &= \frac{1}{C_L - C_R} \left(A(C_L) - A(C_R) - (C_L - C_R) \dot{X}(s) - (II) + (I) + (III) \right). \end{aligned}$$

Then we integrate the above equation in time on $[0, t]$ to get:

$$\begin{aligned} |\sigma t - X(t)| &\leq C \left(t \cdot \max_{s \in (0, t)} \underbrace{\left| A(C_L) - A(C_R) - (C_L - C_R) \dot{X}(s) - (II) \right|}_{(II')} \right. \\ &\quad \left. + \left| \int_0^t (I) ds \right| + t \cdot \max_{s \in (0, t)} \left| (III) \right| \right). \end{aligned} \quad (24)$$

We observe

$$(II') \leq \underbrace{\left| A(C_L) - A(C_R) - \frac{1}{R} \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot A(U) dx \right|}_{(II'_1)} + \underbrace{\left| - (C_L - C_R) \dot{X}(s) + \frac{1}{R} \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot \left(\dot{X}(s) U(s, x) \right) dx \right|}_{(II'_2)}.$$

For (II'_1) , we compute

$$\begin{aligned} (II'_1) &= \left| A(C_L) - \frac{1}{R} \int_{-2R+X(s)}^{-R+X(s)} A(U) dx - A(C_R) + \frac{1}{R} \int_{R+X(s)}^{2R+X(s)} A(U) dx \right| \\ &\leq \frac{1}{R} \left[\int_{-2R+X(s)}^{-R+X(s)} |A(C_L) - A(U)| dx + \int_{R+X(s)}^{2R+X(s)} |A(U) - A(C_R)| dx \right]. \end{aligned}$$

We use $|A(y) - A(z)| \leq C|y - z|$ for $|y|, |z| \leq M_1$ to get

$$\leq \frac{C}{R} \int_{-2R+X(s)}^{2R+X(s)} |U - S| dx.$$

We use the Hölder's inequality and Lemma 3.1 to get

$$(II'_1)^2 \leq \frac{C}{R} \cdot \int_{\mathbb{R}} \eta(U(s)|S(s)) dx.$$

Likewise, for the second term (II'_2) , we have

$$\begin{aligned} (II'_2) &= |\dot{X}(s)| \cdot \left| - (C_L - C_R) + \frac{1}{R} \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot U(s, x) dx \right| \\ &\leq \frac{C}{R} \int_{-2R+X(s)}^{2R+X(s)} |U - S| dx \leq \frac{C}{\sqrt{R}} \cdot \|U(s) - S(s)\|_{L^2(\mathbb{R})} \end{aligned}$$

where we used $|\dot{X}(s)| \leq C$. Thus we get

$$(II')^2 \leq \frac{C}{R} \cdot \int_{\mathbb{R}} \eta(U(s)|S(s)) dx. \quad (25)$$

On the other hand, we compute

$$\begin{aligned} \left| \int_0^t (I) ds \right| &= \left| \int \psi \left(\frac{x - X(t)}{R} \right) \cdot U(t, x) dx - \int \psi \left(\frac{x}{R} \right) \cdot U_0(x) dx \right| \\ &= \left| \int \psi \left(\frac{x - X(t)}{R} \right) \cdot (U(t, x) - S(t, x)) dx + \int \psi \left(\frac{x - X(t)}{R} \right) \cdot S(t, x) dx \right. \\ &\quad \left. - \int \psi \left(\frac{x}{R} \right) \cdot S_0(x) dx - \int \psi \left(\frac{x}{R} \right) \cdot (U_0(x) - S_0(x)) dx \right|. \end{aligned}$$

Note that $\int \psi(\frac{x-X(t)}{R}) \cdot S(t, x) dx = \int \psi(\frac{x}{R}) \cdot S_0(x) dx$. Thus, we have

$$\leq \left| \int \psi(\frac{x-X(t)}{R}) \cdot (U(t, x) - S(t, x)) dx \right| + \left| \int \psi(\frac{x}{R}) \cdot (U_0(x) - S_0(x)) dx \right|.$$

We use the Hölder's inequality and Lemma 3.1 to get

$$\left| \int_0^t (I) ds \right|^2 \leq CR \left(\int_{\mathbb{R}} \eta(U(t)|S(t)) dx + \int_{\mathbb{R}} \eta(U_0|S_0) dx \right). \quad (26)$$

Also, we have

$$\begin{aligned} |(III)| &= \frac{\varepsilon}{R} \left| \int \psi'(\frac{x-X(s)}{R}) \cdot \partial_x U(s, x) dx \right| \\ &= \frac{\varepsilon}{R} \left| \int_{-2R+X(s)}^{-R+X(s)} \partial_x U(s, x) dx - \int_{R+X(s)}^{2R+X(s)} \partial_x U(s, x) dx \right| \\ &\leq \frac{\varepsilon}{R} \cdot 4 \cdot \|U(s)\|_{L^\infty} \leq \frac{C \cdot \varepsilon}{R}. \end{aligned} \quad (27)$$

Finally, by using (23), we combine (25), (26) and (27) with (24) to get, for any $t \in (0, T)$,

$$|\sigma t - X(t)|^2 \leq C \left(\frac{t^2}{R} + R \right) \cdot \left(\int_{\mathbb{R}} \eta(U_0|S_0) dx + \varepsilon \log\left(\frac{1}{\varepsilon}\right) \right) + \frac{C \cdot \varepsilon^2 \cdot t^2}{R^2}.$$

Since the above estimate holds for any $0 < R < \infty$, the estimate (11) follows once we take $R := t^{2/3}$ (changing C^* if needed).

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